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The curve of compactified $6D$ gauge theories and integrable systems

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Abstract: We analyze the Seiberg-Witten curve of the six-dimensional $\mathcal{N} = (1,1)$ gauge theory compactified on a torus to four dimensions. The effective theory in four dimensions is a deformation of the $\mathcal{N} = 2^*$ theory. The curve is naturally holomorphically embedding in a slanted four-torus — actually an abelian surface — a set-up that is natural in Witten's M-theory construction of $\mathcal{N} = 2$ theories. We then show that the curve can be interpreted as the spectral curve of an integrable system which generalizes the N-body elliptic Calogero-Moser and Ruijsenaars-Schneider systems in that both the positions and momenta take values in compact spaces. It turns out that the resulting system is not simply doubly elliptic, rather the positions and momenta, as two-vectors, take values in the ambient abelian surface. We analyze the two-body system in some detail. The system we uncover provides a concrete realization of a Beauville-Mukai system based on an abelian surface rather than a K3 surface.

Keywords: Brane Dynamics in Gauge Theories, Extended Supersymmetry, Supersymmetric Effective Theories, Integrable Equations in Physics.

Contents

1. Introduction

The Coulomb branch of four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry is described at low energy by Seiberg-Witten theory [1]. In essence, there is a complex curve, or Riemann surface, whose moduli vary across the Coulomb branch, and whose period matrix determines the couplings of the low energy effective action. Theories with eight supercharges can be defined in higher dimensions, specifically in five and six dimensions. When compactified on a circle and torus, respectively, where we keep the scale of the compactification space finite, this gives rise to a four-dimensional gauge theory with $\mathcal{N} = 2$ symmetry. Hence, we expect that there is a Seiberg-Witten curve which controls the couplings in the low energy effective action which now depends on the details of the compactification space. In particular, we will consider the $\mathrm{U}(N)$ gauge theory with a massive adjoint hypermultiplet, sometimes called the $\mathcal{N} = 2^*$ theory.

The $\mathcal{N} = 2^*$ theory can be lifted to six dimensions where it is has $\mathcal{N} = (1, 1)$ supersymmetry, although strictly in six dimensions the hypermultiplet is massless. Note that the six-dimensional gauge theory is non-renormalizable, but it may be given an ultra-violet completion as a little string theory: the theory describing five branes in Type II string theory. Once compactified on a torus the resulting theory is an $\mathcal{N} = 2$ $\mathrm{U}(N)$ gauge theory in four dimensions with a complexified coupling τ given by g_6^{-2} times the complexified Kähler class of the torus. The effective four-dimensional theory will also depend on the complex structure of the torus ρ and, in addition, one can include a certain twist in the R-symmetry group which gives a mass m to the hypermultiplet in four dimensions. By a twist, we mean that there are non-trivial holonomies in a $\mathrm{U}(1) \subset \mathrm{spin}(4)$ of the R-symmetry group [2]–[4]. So the effective four-dimensional theory will include the fields of the $\mathcal{N} = 2^*$ theory ($\mathcal{N} = 2$ plus massive adjoint hypermultiplet) plus all the higher Kaluza-Klein modes on the torus.

The low-energy effective action of the four-dimensional theory will be described by Seiberg-Witten theory and in particular we focus on the Seiberg-Witten curve. We show how it may be obtained by using a generalization of Witten's brane configuration in Type

IIA/M-theory [5]. The resulting curve is identical, as is expected, to that proposed in [4] related to the moduli space of instantons on a non-commutative 4-torus. The curve has also been determined in [6] via a number of different methods: a Dijkgraaf-Vafa matrix model, geometric engineering and the instanton calculus.

Once we have established the form of the curve, we turn to the question of whether there is a related integrable system. The motivation is as follows. If we consider the $\mathcal{N}=2^*$ theory strictly in four dimensions then the Seiberg-Witten curve is the spectral curve of the complexified N-body elliptic Calogero-Moser system, where the position coordinates take values on an auxiliary torus whose complex structure is τ, the complexified coupling of the four-dimensional theory, and whose momenta are valued in $\mathbf{C}$. If we then lift this theory to five dimensions compactified on a circle, then the Seiberg-Witten curve becomes the spectral curve of the N-body elliptic Ruijsenaars-Schneider integrable system [7, 8, 9]. This is sometimes described as the "relativistic" Calogero-Moser model since the momenta are now periodic in one direction on $\mathbf{C}$ — as one would expect for a rapidity. The "speed of light" is proportional to the inverse of size of the circle so that when the radius goes to zero the Calogero-Moser system is recovered. Now imagine lifting the five-dimensional theory to six dimensions compactified on a torus. The question then is: what is the next object in the chain?

$$\begin{array}{ccccc} \text{Calogero-Moser} & \longrightarrow & \text{Ruijsenaars-Schneider} & \longrightarrow & ? \\ q\in\mathbf{T}^2\,,\quad p\in\mathbb{C} & & q\in\mathbf{T}^2\,,\quad p\in\mathbb{R}\times\mathbf{S}^1 & & q\in\mathbf{T}^2\,,\quad p\overset{?}{\in}\mathbf{T}^2\,. \end{array}$$

One naturally expects that the momenta now take values on a torus, as indicated above. In fact the natural guess is the torus dual to the compactification torus. Hence, with both the positions and the momenta being doubly-periodic the hypothetical integrable system is sometimes called the "doubly-elliptic", or Dell, system. Such a Dell system, in the case of 2 particles, has been defined and investigated in [10]–[12]. It is one of main aims of this paper to find the hypothetical integrable system for any number of particles. Our approach makes prominent a polarized abelian surface with moduli τ, ρ and m, and the system we uncover provides a concrete realization of a Beauville-Mukai system based on the abelian surface, rather than the more often discussed K3 surface. We will compare our system with Dell system in due course.

2. The curve from M theory

There are several ways to find the form of the Seiberg-Witten curve that describes compactified six-dimensional gauge theories. In the first place, the form of the curve follows from the work of [4]. In this work, it was shown how the curve, which we denote Σ, plays a rôle in the description of the moduli space of N instantons in a U(1) gauge theory on a non-commutative torus.[1] Another way to engineer the curve is via a Dijkgraaf-Vafa matrix model, although in this case, the model is actually a two-dimensional matrix field theory

[1] The case of N instantons in a $\mathrm{U}(k)$ gauge theory is relevant to a quiver generalization of the $\mathcal{N}=2^*$ theory where the gauge group becomes $\mathrm{U}(N)^k$; this will be described elsewhere.

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defined on the compactification torus [6]. The matrix model describes a deformation of the four-dimensional theory from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. If the deformation is suitably generic then varying it allows the vacuum to track across the Coulomb branch of the $\mathcal{N} = 2$ theory. In this way, the Seiberg-Witten curve itself can be extracted. Yet another way to engineer the curve is to use equivariant localization techniques, pioneered in [13, 14], to calculate the instanton partition function from which the Seiberg-Witten curve can be extracted from a saddle-point analysis. Going from the four-dimensional theory to the compactified six dimensional one involves replacing the integral over instanton moduli space by the partition function of a two-dimensional sigma model whose target is the same space. The details are described in [6].

In this section, we take a different approach by proposing a form for the curve based on Witten's M-theory construction of $\mathcal{N} = 2$ theories [5]. The answer agrees with other methods. We start with the Type IIA configuration which describes the four-dimensional $\mathcal{N} = 2^*$ theory. There is one NS5-brane whose world-volume spans $\{x^0, x^1, x^2, x^3, x^4, x^5\}$ and N D4-branes spanning $\{x^0, x^1, x^2, x^3, x^6\}$. The direction x^6 is periodic. The position of each D4-brane in $x^{4,5}$ may be described by the complex variable $x = x^4 + ix^5$. In order to incorporate the mass m the spacetime must be modified: as one goes around the x^6 circle x shifts by m.

As explained in [5], after lifting to M-theory the configuration is described by a single M5-brane whose world-volume fills $\{x^0, x^1, x^2, x^3\}$ while the remaining directions are described by a Riemann surface Σ embedded in the four-dimensional space $\mathfrak{Q}$ whose coordinates are $\{x^4, x^5, x^6, x^{10}\}$. The M-theory direction x^{10} together with x^6 are valued on a torus of complex structure τ, the coupling constant of the four-dimensional theory. We will introduce the holomorphic coordinate

$$z = \frac{1}{2\pi R_{10}} \left(x^{10} + ix^6\right), \tag{2.1}$$

where $2\pi R_{10}$ is the circumference of the M-theory circle, and define

$$\mathbf{T}^2_z = \left\{z \in \mathbb{C} \big|\ z \sim z + p + \tau q\,, \qquad p, q \in \mathbb{Z}\right\}. \tag{2.2}$$

In order to incorporate the mass m, the complex x-plane is then fibred over this base torus so that as one goes around the B-cycle of $\mathbf{T}^2_z$ there is a shift $x \to x + m$. This means that, although $\mathfrak{Q}$ is equal to $\mathbf{R}^2 \times \mathbf{T}^2_z$ locally, this is not true globally.

The curve $\Sigma \hookrightarrow \mathfrak{Q}$ describing the geometry of the M5-brane is the Seiberg-Witten curve of the four-dimensional theory. The curve can be described algebraically via

$$F(z, x) = x^N + f_1(z)x^{N-1} + f_2(z)x^{N-2} + \cdots + f_N(z) = \prod_{i=1}^{N}(x - x_i(z)) = 0\,. \tag{2.3}$$

The coefficients are fixed by the following conditions. First, in order to incorporate the mass m, we need

$$F(z + 1, x) = F(z, x)\,, \qquad F(z + \tau, x) = F(z, x + m)\,. \tag{2.4}$$

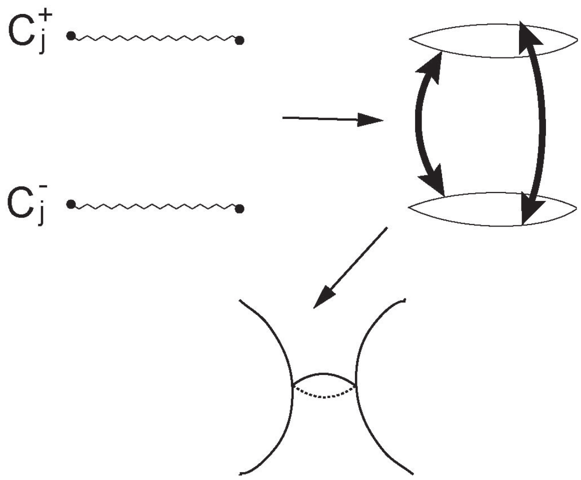

Figure 1: Each pair of cuts $\mathcal{C}_j^- = (a_j, b_j)$ and $\mathcal{C}_j^+ = (a_j + m, b_j + m)$ are identified in such a way as to form a handle on the x-plane.

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The eigenvalues $x_i(z)$ define a branched N-fold cover of the base torus $\mathbf{T}_z^2$. There is a distinguished point, which we choose at $z = 0$, that corresponds to the position of the NS5-brane. At $z = 0$ on exactly one of the sheets, say i, a point on Σ which we denote as P_0, the associated function $x_i(z)$ has a simple pole with residue m. Note that we can define the meromorphic function v on Σ which takes the form[2]

$$v_j(z) = N x_j(z) - m \left(\zeta(z) - 2z\zeta\left(\frac{1}{2}\right)\right) \tag{2.5}$$

on the j^{th} sheet. Although v is single-valued on Σ it now has simple poles on each sheet at $z = 0$, with residues $-m$, on sheets $j \neq i$, and with residue $(N-1)m$ on the i^{th} sheet. The genus of Σ follows from the Riemann-Hurwitz Theorem and is found to be N. In addition, the number of moduli of Σ can be computed using the Riemann-Roch Theorem; one finds N.

We can view the curve Σ as N copies of the base torus $\mathbf{T}_z^2$ plumbed together by $N-1$ branch cuts. The A and B cycles on the base torus $\mathbf{T}_z^2$ lift to a basis of 1-cycles A_i and B_i, $i = 1, \ldots, N$, on Σ. Let ω_i be the associated abelian differentials of the first kind. We can then identify z as a multi-valued function on Σ in the following way. First of all, z must shift by 1 around each of the A_i cycles; this means

$$z(P) = \int_{P_0}^{P} \sum_{j=1}^{N} \omega_j \,, \tag{2.6}$$

where $P \in \Sigma$ and P_0 is some fixed origin. Around each B_i cycle z must shift by τ, the coupling. Hence, we have the following constraint on the period matrix Π of Σ:

$$\sum_{j=1}^{N} \Pi_{ij} = \tau \qquad \forall i \,. \tag{2.7}$$

[2]In the following $\zeta(z)$ is the Weierstrass ζ-function with half periods $1/2$ and $\tau/2$.

Another way to view the surface is in terms of the multi-valued function x. This is continuous around the A_j cycles but jumps by m around the B_j cycles. The function x covers the complex plane with the point at infinity corresponding to the distinguished point P_0, the position of the NS5-brane. There are N pairs of cuts $\mathcal{C}_j^{\pm}$ with (complex) end-points (a_j, b_j) and $(a_j + m, b_j + m)$, $j = 1, \ldots, N$. The cuts in a given pair are glued together so that the top/bottom of the upper cut is glued to the bottom/top of the lower cut. Thus each pair generates a handle as illustrated in figure 1. The moduli correspond to positions of the ends of the lower cuts, so number $2N$; however, there are N conditions (2.7) on the period matrix so, all-in-all, there are N (complex) moduli.

Now we consider the six-dimensional theory compactified on the torus $\mathbf{T}^2$ defined as follows,

$$\mathbf{T}^2 = \Big\{ y \in \mathbb{C} \,\Big|\, y \sim y + \frac{\beta}{\rho - \bar{\rho}}(p + \rho q) \,, \qquad p, q \in \mathbb{Z} \Big\} . \tag{2.8}$$

In order to describe the compactified six-dimensional gauge theory in the M-theory set-up we have to compactify $\{x^4, x^5\}$ on the dual to the compactification torus defined as

$$\tilde{\mathbf{T}}^2 = \Big\{ y \in \mathbb{C} \,\Big|\, y \sim y + \frac{2\pi i}{\beta}(p + \rho q) \,, \qquad p, q \in \mathbb{Z} \Big\} . \tag{2.9}$$

So the space $\mathfrak{Q}$ in which Σ is embedded becomes a 4-torus. In order to incorporate the mass m, it cannot be simply the product $\tilde{\mathbf{T}}^2 \times \mathbf{T}^2_z$. Rather as one goes around the B-cycle of the $\mathbf{T}^2_z$, x must shift by m, as in the four-dimensional theory. This shift must be mirrored in the opposite direction. This means that $\mathfrak{Q}$ must be a "slanted" product of the two tori. We now argue that the form of $\mathfrak{Q}$ is determined by the condition that there exists a curve Σ embedded in it with the appropriate properties. As a real manifold we can think of $\mathfrak{Q}$ as $\mathbf{R}^4/\Lambda$, where Λ is a rank 4 lattice. If a set of basis vectors for Λ are λ_α, $\alpha = 1, \ldots, 4$, then we can think of $\mathfrak{Q}$ as the region

$$\sum_{\alpha=1}^{4} y_\alpha \lambda_\alpha \in \mathbb{R}^4 \tag{2.10}$$

parameterized by four real variables $0 \leq y_\alpha \leq 1$ whose faces $y_\alpha = 0$ and $y_\alpha = 1$ are identified. We propose that a curve Σ of the right form exists when $\mathfrak{Q}$ is an abelian surface (a 2-dimensional abelian variety) [15, 16]. To start, we view $\mathbb{R}^4$ as a complex manifold $\mathbb{C}^2$ and introduce holomorphic coordinates (z_1, z_2). These are related to the real coordinates y_α via the 2×4 period matrix

$$z_i = \sum_{\alpha=1}^{4} \Omega_{i\alpha} y_\alpha \,. \tag{2.11}$$

In order to be an abelian variety, the condition that it can be embedded in projective space, Ω must satisfy certain conditions (the Riemann conditions). With a suitable choice of basis for Λ it can be shown that Ω may be put in the form

$$\Omega = \begin{pmatrix} \delta_1 & 0 & \Gamma_{11} & \Gamma_{12} \\ 0 & \delta_2 & \Gamma_{21} & \Gamma_{22} \end{pmatrix} , \tag{2.12}$$

where $\delta_{1,2}$ are integers and $\delta_1 | \delta_2$. Then the conditions that $\mathfrak{Q}$ be an abelian variety are that the 2×2 matrix Γ is symmetric and $\mathrm{Im}\,\Gamma$ is positive definite. In this basis, (δ_1, δ_2) defines the polarization of $\mathfrak{Q}$. The conditions on Γ ensures that

$$\omega = \delta_1\, dy_1 \wedge dy_3 + \delta_2\, dy_2 \wedge dy_4 \,, \tag{2.13}$$

is a $(1,1)$-form. The fact that $\mathfrak{Q}$ is an abelian surface will ensure that there exists a holomorphic curve Σ with the right properties. This curve arises in the following way. A polarized abelian variety has an associated line bundle $\mathcal{L}$ whose $1^{\rm st}$ Chern class $c_1(\mathcal{L}) = [\omega]$. This line bundle admits $\delta_1\delta_2$ holomorphic sections which are constructed in terms of generalized theta functions. Our conventions for these latter functions are

$$\Theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (Z|\,\Pi) = \sum_{m\in\mathbb{Z}^g} \exp\left(\pi i(m+\delta)\cdot \Pi \cdot (m+\delta) + 2\pi i (Z+\epsilon)\cdot(m+\delta)\right) . \tag{2.14}$$

Here Z, δ, ϵ and m are g-vectors and Π is a $g\times g$ matrix (with $\mathrm{Im}\,\Pi$ positive definite). The basis for the holomorphic sections of $\mathcal{L}$ are then given by the $\delta_1\delta_2$ theta functions ($g = 2$ in the case of an abelian surface)

$$\Theta \begin{bmatrix} i/\delta_1 & j/\delta_2 \\ 0 & 0 \end{bmatrix} (z_1\ z_2|\Gamma) \qquad 0 \le i < \delta_1 \,, \qquad 0 \le j < \delta_2 \,. \tag{2.15}$$

A general section is then a linear combination of the above. The divisor of a given section naturally defines a curve Σ in the homology class dual to $c_1(\mathcal{L}) = [\omega]$. In other words, for some coefficients A_{ij}, the curve Σ is defined by

$$\sum_{i=0}^{\delta_1-1}\sum_{j=0}^{\delta_2-1} A_{ij}\Theta \begin{bmatrix} i/\delta_1 & j/\delta_2 \\ 0 & 0 \end{bmatrix} \big(z_1\ z_2|\Gamma\big) = 0 \,. \tag{2.16}$$

In order to realize the structure above in the M-theory scenario we must relate the coordinates (z_1, z_2) to the physical coordinates (z, x). Notice that if $\Gamma_{12} = \Gamma_{21} = 0$ then $\mathfrak{Q}$ is the product of two elliptic curves. This will describe the degenerate limit when the mass $m = 0$ and in this limit the two elliptic curves are identified with $\mathbf{T}^2_z$ and $\tilde{\mathbf{T}}^2$. In other words, in this limit, we identify the fundamental domain of $\mathbf{T}^2_z$ with $z = y_1 + y_3\tau$ and that of $\tilde{\mathbf{T}}^2$ with $x = (2\pi i/\beta)(y_2 + y_4\rho)$. The next fact we need is that the curve Σ we are after must be wrapped N times around the z-torus but only once around the x-torus. This requires some explanation. Suppose β is small so that the dual torus $\tilde{\mathbf{T}}^2$ is very large. In this limit it makes sense to think of the curve as approximately equal to the four-dimensional one. The four-dimensional curve goes off to infinity in the x-plane in the neighbourhood of the distinguished point P_0. In the compactified theory this region must wrap the torus once. The multiple wrapping situation will occur in the quiver theory generalization of our set-up and will be described elsewhere.

So more precisely, the homology class of Σ should be Poincaré dual to the form ω in (2.13) with $\delta_1 = 1$ and $\delta_2 = N$. In the limit $m = 0$, this fixes

$$\Omega\big|_{m=0} = \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & N & 0 & N\rho \end{pmatrix} . \tag{2.17}$$

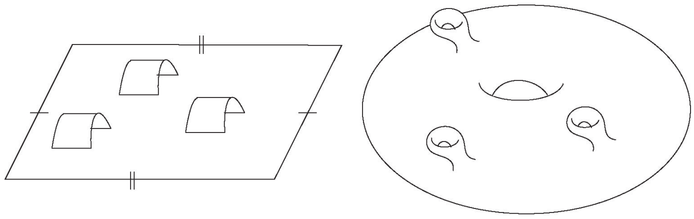

Figure 2: On the left, the surface Σ realized as the cut x-torus $\tilde{\mathbf{T}}^2$. The cuts in each of the N pairs are separated by m and are glued together as in figure 1. On the right, an impression of the surface realized as N handles $\tilde{\mathbf{T}}^2$.

A non-zero mass m corresponds to a non-vanishing off-diagonal component Γ_{12}. Given that x must shift by m as one goes around the B cycle of the z-torus, determines the period matrix Γ for non-vanishing mass:

$$\Gamma = \begin{pmatrix} \tau & \frac{N\beta m}{2\pi i} \\ \frac{N\beta m}{2\pi i} & N\rho \end{pmatrix} \tag{2.18}$$

and in addition we discover

$$z_1 = z\,, \qquad z_2 = \frac{N\beta x}{2\pi i}\,. \tag{2.19}$$

With this period matrix, the curve Σ in (2.16) has genus $N+1$. The coefficients A_{0j}, along with a choice of origin for Z, are the moduli of the curve. Since the A_{0j} are defined up to an overall complex re-scaling they parameterize a copy of $\mathbf{P}^{N-1}$. Hence the moduli space of the curve is $\mathfrak{M} = \mathfrak{Q} \times \mathbf{P}^{N-1}$, which is $N+1$ complex dimensional.

We can picture the curve in two ways. First, as in the four-dimensional case, as the x-plane, which is now a torus with periods $(2\pi i/\beta)(1,\rho)$, together with N pairs of cuts across which x jumps by m and whose edges are identified to create a handle, as in figure 1. This is illustrated in figure 2. The second representation consists of N copies of a torus in the z-plane, with periods $(1,\tau)$, and which are joined by $N-1$ branch cuts. On the face of it, such a surface would have genus N but on one of the sheets there is a pair of cuts across which z jumps by $N\beta m/(2\pi i)$ whose edges are identified to create an extra handle. This is illustrated in figure 3.

Using the explicit expression for the theta-function, the form of the curve (2.16) can be re-cast in the following way which makes the reduction to five and four dimensions more immediate:

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{m}{2\pi i}\right)^n \partial_z^n \theta_1(\pi z|\tau)\partial_x^n H(x) = 0\,, \tag{2.20}$$

where θ_1 is the usual Jacobi theta function related to Θ for $g=1$ via

$$\theta_1(\pi z|\tau) = -\Theta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(z|\tau)\,. \tag{2.21}$$

In the above,

$$H(x) = \prod_{j=1}^{N} \theta_1\left(\frac{\beta}{2i}(x-\zeta_j)|\rho\right)\,. \tag{2.22}$$

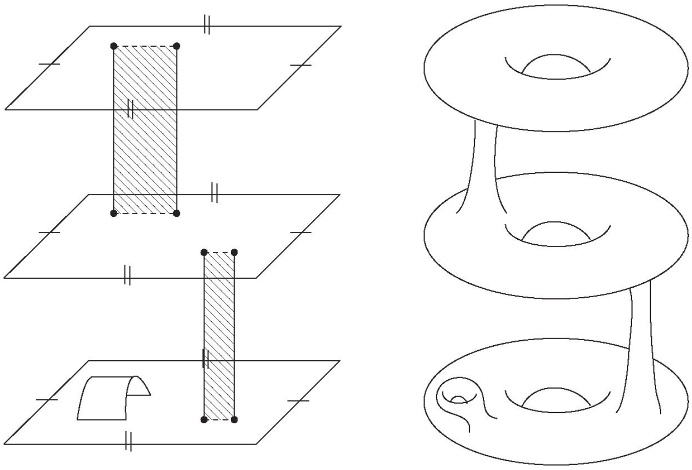

Figure 3: On the left, the surface Σ realized as N copies of the z-torus $\mathbf{T}^2_z$ connected by $N-1$ branch cuts. On one of the sheets there is an additional pairs of cuts separated by $N\beta m/(2\pi i)$ which are glued together as in figure 1. On the right, is an impression of the surface illustrating the N copies of the z-torus plumbed together along with the additional handle on one of the sheets.

Here, ζ_j are N of the moduli and the remaining one corresponds to shifting z by a constant. By equating (2.20) and (2.16), up to some constant shifts in x and z, we deduce that

$$A_{0j}(\zeta_i) = \sum_{\{i_k\}} \Theta \begin{bmatrix} i_1/N & \cdots & i_{N-1}/N \\ 0 & \cdots & 0 \end{bmatrix} \left(\frac{N\beta}{2\pi i}(\zeta_2 - \bar{\zeta}) \ \cdots \ \frac{N\beta}{2\pi i}(\zeta_N - \bar{\zeta}) \middle| \tilde{\Pi} \right), \tag{2.23}$$

where the j-dependence arises through the definition of the sum which is over $i_k = 0, \ldots, N-1$ subject to

$$j + \sum_{k=1}^{N-1} i_k \in N \cdot \mathbb{Z} \tag{2.24}$$

and $\tilde{\Pi}$ is an $(N-1) \times (N-1)$-dimensional matrix with elements

$$\tilde{\Pi}_{ij} = \rho N (N\delta_{ij} - 1) \,. \tag{2.25}$$

Finally

$$\bar{\zeta} = \frac{1}{N} \sum_{i=1}^{N} \zeta_i \,. \tag{2.26}$$

To go from the curve of the six-dimensional theory that of the five-dimensional theory, one takes $\rho \to i\infty$ in which case

$$H(x) \to \prod_{j=1}^{N} \sinh\left(\frac{\beta}{2}(x - \zeta_i) \right), \tag{2.27}$$

and from the five to the four-dimensional theory one takes $\beta \to 0$ giving rise to

$$H(x) \to \prod_{j=1}^{N} (x - \zeta_i) \,. \tag{2.28}$$

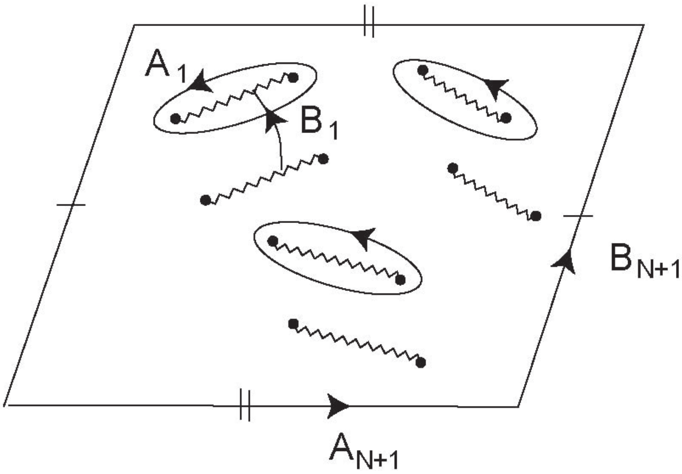

Figure 4: The surface Σ realized as the cut torus $\tilde{\mathbf{T}}^2$ showing our choice of homology basis.

The curve of the four-dimension theory is identical to the curve described by Donagi and Witten [17]. It is well-known that this is the spectral curve of the N-body elliptic Calogero-Moser integrable system [18, 19, 20]. The curve of the five-dimensional theory can be shown to be the spectral curve of the Ruijsenaars-Schneider integrable system [9] as predicted by Nekrasov [7]. The weak and strong coupling limits of these theories have been investigated in [21].

Both x and z are multi-valued functions on the curve while dx and dz are holomorphic differentials (abelian differentials of the 1st kind) which we can identify, in terms of the basis $\{\omega_i\}$, as

$$dx = \frac{2\pi i}{\beta}\omega_{N+1}\,, \qquad dz = \sum_{i=1}^{N}\omega_i\,, \tag{2.29}$$

with respect to a homology basis $\{A_i, B_i\}$. Here, the A_i cycles, $i = 1,\ldots,N$, encircle the cuts $[a_i + m, b_i + m]$ on the x-torus and where B_i, $i = 1,\ldots,N$, join the bottom cut to the top cut in each pair (and hence because of the gluing condition are cycles). This leaves A_{N+1} and B_{N+1} which are the usual A and B cycles on the x-torus $\tilde{T}^2$. The cycles are shown in figure 4. The abelian differentials of the 1st kind are then normalized by $\oint_{A_i}\omega_j = \delta_{ij}$ and given this $\oint_{B_i}\omega_j = \Pi_{ij}$. The fact that z is valued on the torus $\mathbf{T}^2_z$ means that, as in (2.7), there is a condition on the period matrix of Σ:

$$\sum_{j=1}^{N}\Pi_{ij} = \tau \qquad \forall i\,. \tag{2.30}$$

In addition, since x jumps by m around the cycles B_i, $i = 1,\ldots,N$, and by $2\pi i\rho/\beta$ around the remaining cycle B_{N+1}, the full period matrix of Σ has the form

$$\Pi = \begin{pmatrix} \Pi_{11} & \cdots & \Pi_{1N} & \beta m/2\pi i \\ \vdots & \ddots & \vdots & \vdots \\ \Pi_{N1} & \cdots & \Pi_{NN} & \beta m/2\pi i \\ \beta m/2\pi i & \cdots & \beta m/2\pi i & \rho \end{pmatrix}, \qquad \sum_{j=1}^{N}\Pi_{ij} = \tau\,. \tag{2.31}$$

Notice that z jumps by

$$\oint_{B_{N+1}} \sum_{i=1}^{N} \omega_i = \frac{N\beta m}{2\pi i}\,, \tag{2.32}$$

around B_{N+1} a fact that is consistent with the period matrix of $\mathfrak{Q}$ in (2.18).

There is yet another way to write the curve. First of all, one introduces another genus $N+1$ curve $\hat{\Sigma}$ with a period matrix

$$\hat{\Pi} = \begin{pmatrix} \tau & \beta m/2\pi i & \beta m/2\pi i & \cdots & \beta m/2\pi i \\ \beta m/2\pi i & \rho & 0 & \cdots & 0 \\ \beta m/2\pi i & 0 & \rho & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \beta m/2\pi i & 0 & \cdots & 0 & \rho \end{pmatrix}. \tag{2.33}$$

This curve is an N-fold unbranched cover of a genus 2 curve. It is then not too difficult to show that (2.20) is equivalent to

$$\Theta \begin{bmatrix} 1/2 & \cdots & 1/2 \\ 1/2 & \cdots & 1/2 \end{bmatrix} \left(z,\, \frac{N\beta}{2\pi i}(x-\zeta_1),\, \cdots,\, \frac{N\beta}{2\pi i}(x-\zeta_N) \big| \hat{\Pi} \right) = 0 \tag{2.34}$$

up to constants shifts in x and z. The expression (2.16) arises when reducing the theta function of $\hat{\Sigma}$ in terms of the genus two curve and the associated Prym variety. We see that the curve Σ lies in the theta divisor of $\hat{\Sigma}$. This means that there are N points $\hat{P}_i$ on $\hat{\Sigma}$ such that

$$z = \sum_{i=1}^{N} \int_{\hat{P}_0}^{\hat{P}_i} \hat{\omega}_0 + \hat{\mathcal{K}}_0\,, \qquad \frac{N\beta}{2\pi i}(x-\zeta_j) = \sum_{i=1}^{N} \int_{\hat{P}_0}^{\hat{P}_i} \hat{\omega}_j + \hat{\mathcal{K}}_j \quad j = 1, \ldots, N\,, \tag{2.35}$$

where we have absorbed constant shifts in z and x and $\mathcal{K}_j$ is the vector of Riemann constants. So given $x - \zeta_i$, $i = 1, \ldots, N$, the last N equations determine the $\hat{P}_i$ which then determines z via the first equation.

3. An integrable system

It is well-known that Seiberg-Witten curves are the spectral curves of integrable systems. For example, the curve of the four-dimensional theory is the spectral curve of the N-body elliptic Calogero-Moser system. This is more than a happy coincidence. In order to see this, it is helpful to take the theory in four dimensions, in this case the $\mathcal{N} = 2^*$ theory, and compactify it to three dimensions on a circle of finite radius [22]. If the radius is very large compared with the scale of symmetry breaking, then it is appropriate first of all to consider the low-energy effective theory in four dimensions, in this case a $\mathrm{U}(1)^N$ gauge theory with gauge couplings given by the period matrix of the Seiberg-Witten curve,

$$\mathcal{L}_{\text{eff}} = \sum_{i,j=1}^{N} \Big[\mathrm{Im}(\Pi_{ij}) F_i \wedge {}^*F_j + i\mathrm{Re}(\Pi_{ij}) F_i \wedge F_j \Big] \ldots\,, \tag{3.1}$$

compactified on a circle to three dimensions. The degrees-of-freedom in three dimensions consist of the Wilson lines of the $\mathrm{U}(1)^N$ gauge field around the circle along with the three-dimensional gauge fields which may be dualized into scalars. The Wilson lines and dual photons naturally combine into N scalar fields which take values on a complex torus with period matrix determined by the couplings Π_{ij}; in other words the effective degrees-of-freedom are described by a point in the jacobian $\mathfrak{J}(\Sigma)$ of Σ. In addition to this, we also have the moduli of the curve itself. So, all-in-all, the moduli space of vacua of the three-dimensional theory consists of a fibration of $\mathfrak{J}(\Sigma)$ over $\mathfrak{M}(\Sigma)$. This is actually the phase space of a complexified integrable system, where the fibre corresponds to the angle variables $\{\theta_i\}$ and base space is parameterized by the conjugate action variables $\{a_i\}$. As yet we don't have an explicit expression for the action variables but this will emerge. In this context Σ, which is fixed under evolution in the integrable system, is called the spectral curve. The reason why the compactification to three dimensions is useful is because the symplectic form (which in the holomorphic context is a $(2,0)$-form) is independent of the compactification radius and this independence means that various holomorphic quantities that can be calculated in the three-dimensional theory are actually valid in the four-dimensional theory. In particular, we have in mind the vacuum structure after breaking to $\mathcal{N}=1$ [23]–[26]. The above connection with integrable systems is part of a more general setting. The moduli space of the Seiberg-Witten curve, parameterized by the action variables of the integrable system, is a special Kähler manifold. Indeed, special Kähler geometries arise as moduli spaces in many examples of interest (for example, the scalars of $\mathcal{N}=2$ four-dimensional Einstein-Maxwell super-gravities [27], or appropriate CFT's [28, 29], or CY_3's [30]). Freed [31] has argued that the cotangent bundle to such, with appropriate restrictions, is the phase space of (an algebraically) completely integrable system.

In the above discussion the integrable system appears directly in terms of action/angle variables and one can ask the question as to whether there is a representation in terms of more familiar dynamical variables such as the positions and momenta of particles. It is at this point that the realization of the curve in the physical problem to hand becomes important. The point is that in the four-dimensional theory the curve naturally appears as holomorphically embedded in a four-dimensional space $\mathfrak{Q}$ which we recall for our four-dimensional theory is (in terms of real geometry) locally of the form $\mathbf{T}^2_z\times\mathbb{R}^2$. We shall soon see that this embedding provides the position and momentum basis. The curve (2.3) (or equivalently (2.20) with $H(x)$ as in (2.28)) has $N+1$ moduli corresponding to the ζ_i, $i=1,\dots,N$, along with the choice of origin for z. The origin for z is fixed by choosing the position of the NS5-brane to be $z=0$ as in section 2. Note that the choice of origin for x is incorporated in the average $\sum_{i=1}^N\zeta_i/N$; we choose $\sum_{i=1}^N\zeta_i=0$. This leaves $N-1$ of the moduli to vary giving a subspace $\mathfrak{M}_0\subset\mathfrak{M}(\Sigma)$. These moduli are parameterized by $N-1$ action variables which are the canonically conjugate variables to an $N-1$ dimensional subspace of the jacobian $\mathfrak{J}_0\subset\mathfrak{J}(\Sigma)$ defined by the condition $\sum_{i=1}^N\theta_i=0$. (Recall that in the four-dimensional case the curve Σ has genus N.)

We now show that the remaining non-trivial part of the phase space consisting of $N-1$ action and $N-1$ angle variables are bi-rationally equivalent to the degrees-of-freedom of $N-1$ points in the ambient space $\mathfrak{Q}$ with local coordinates $Z_i=(q_i,p_i)$ in $\mathbf{T}^2_z\times\mathbb{R}^2$. The equivalence is straightforward to describe. Generically at least, there is a unique curve Σ in the reduced moduli space (of dimension $N-1$) which goes through $N-1$ points in $\mathfrak{Q}$. This curve naturally has $N-1$ marked points on it P_j, $j=1,\ldots,N-1$. These points map to a point in the reduced jacobian $\mathcal{J}_0$ by first mapping into the jacobian $\mathcal{J}(\Sigma)$ via the Abel map:

$$\theta_i = \sum_{j=1}^{N-1} \int_{P_0}^{P_j} \omega_i \,, \tag{3.2}$$

where P_0 is a fixed base point, and then by using the projection $\mathcal{J}(\Sigma) \to \mathcal{J}_0$. It is straightforward to follow the map in the other direction. The fact that the equivalence is only bi-rational refers to the fact that the map may break down at certain non-generic points and in particular when some of the points (q_i, p_i) coincide.

One can go on to show that the bi-rational equivalence between the positions and momenta and the angle and action variables (where the latter are as yet unspecified) is a canonical transformation. We will do this for the six-dimensional case later. The resulting description of the integrable system in terms of $N-1$ points in $\mathfrak{Q}$ is exactly what one expects for the N-body elliptic Calogero-Moser system. In this case, the $N-1$ quantities q_i and p_i represent the relative positions of the N particles. So the projection onto $\mathfrak{M}_0$ and $\mathcal{J}_0$ simply removes the trivial centre-of-mass motion.

The above picture works in the same way for the five-dimensional theory. In this case, $\mathfrak{Q}$ is locally $\mathbf{S}^1 \times \mathbb{R} \times \mathbf{T}^2_z$ and, in particular, the "momenta" are now valued on $\mathbf{S}^1 \times \mathbf{R}$, as a real manifold. This is appropriate for a relativistic system where the βp_i play the rôle of the rapidities. It is indeed one way of describing the Ruijsenaars-Schneider system, as the relativistic generalization of Calogero-Moser. Once again, the reduced system describes the relative motion of an N-body system.

We now turn to the six-dimensional theory. One's first thought is that the six-dimensional theory will require the momenta to take values on a torus, in fact the dual torus $\tilde{\mathbf{T}}^2$. So with the positions taking values on the torus $\mathbf{T}^2_z$, this leads one to expect the existence of the so-called "doubly elliptic", or Dell, integrable system [10]–[12]. However, we note that the mass parameter m evident in our description of the ambient space $\mathfrak{Q}$ means that in general we are not dealing simply with a global product of two tori.[3] Thus the system we construct is more than "doubly elliptic". We shall compare our construction with that of Dell system shortly.

The main proposition of this paper is that even in the six-dimensional theory, Σ is the spectral curve of a completely integrable mechanical system, albeit of a rather unusual kind. In particular, the phase space of the system can be viewed, as in the four-dimensional case, as the moduli space of the curve Σ, with the jacobian $\mathcal{J}(\Sigma)$ fibered over it. In this section, we show that the integrable system can be thought of as describing the interactions of a set of "particles" whose whose positions and momenta (q,p), as a 2-vector, take values that are the local coordinates of a point on the abelian surface $\mathfrak{Q}$. The fact that the momenta take values in a compact space means that the analogy with a set of particles is not to be

[3]We note that for suitable rational values of $\beta m/2\pi i$ we note that $\mathfrak{Q}$ does however take on this product structure.

taken too literally. However, once the x-torus decompactifies in one, or both, directions, so the curve reduces to the five and four-dimensional one, respectively, then the system really can be interpreted in terms of a system of particles.

Much of following is identical to the four-dimensional case, but there are some differences. Loosely speaking the phase space of the integrable system is identified with the fibre bundle whose base is the moduli space of the curve Σ and whose fibre is the jacobian of the curve $\mathfrak{J}(\Sigma)$. However, in the six-dimensional case the separation of the "centre-of-mass" factors is more involved. Recall that the moduli space of the curve, $\mathfrak{M}(\Sigma)$, and hence the base of the fibration, can be identified as the product

$$\mathfrak{M}(\Sigma) \simeq \mathfrak{Q} \times \mathbf{P}^{N-1} \tag{3.3}$$

which has (complex) dimension $N+1$. The total space of the bundle consequently has the form $\mathfrak{Q} \times \mathbf{X}$, where $\mathbf{X}$ is a fibering of $\mathfrak{J}(\Sigma)$ over $\mathbf{P}^{N-1}$. From the embedding of $\Sigma \hookrightarrow \mathfrak{Q}$ we may pull back the differentials dz and dx on $\mathfrak{Q}$ to abelian differentials of the 1^{st} kind on Σ (2.29). This gives us a natural embedding of the abelian surface $\iota : \mathfrak{Q} \hookrightarrow \mathfrak{J}(\Sigma)$. This embedding means that there is an abelian subvariety $\mathfrak{J}_0$ of $\mathfrak{J}(\Sigma)$ such that $\mathfrak{Q} \oplus \mathfrak{J}_0$ is isogenous to $\mathfrak{J}(\Sigma)$. Hence, $\mathbf{X}$ can be viewed as a fibered product of the fixed $\mathfrak{Q}$ and $\mathbf{Y}$, where $\mathbf{Y}$ is a complex $2N-2$ dimensional space. The space $\mathbf{Y}$ itself a non-trivial fibering of the abelian subvariety $\mathfrak{J}_0 \subset \mathfrak{J}(\Sigma)$ over the reduced moduli space $\mathfrak{M}_0 = \mathbf{P}^{N-1}$.

Before delving into the technicalities, the idea that will emerge is that the space $\mathbf{X}$ is bi-rationally equivalent to the symmetric product $\text{Sym}^N(\mathfrak{Q})$ [32]–[34] (see also [35]). We can equivalently think of this as the Hilbert scheme $\text{Hilb}^{[N]}\mathfrak{Q}$. If we take local coodinates $(q,p) = (z,x)$ on $\mathfrak{Q}$, then the N points (q_i, p_i) define the positions and "momenta" of N particles. Separating out the centre-of-mass motion, leaves us with the $N-1$ relative positions and momenta and this corresponds to the reduced phase space that we denoted $\mathbf{Y}$ above.

Now we will identify more explicitly the reduced phase $\mathbf{Y}$ and show that it is bi-rationally equivalent to the centred motion of N points in $\mathfrak{Q}$. Let us denoted the coordinates in $\mathfrak{J}(\Sigma)$ by θ_i, $i = 1, \ldots, N+1$. We may describe the embedding of the abelian surface $\iota(\mathfrak{Q}) \hookrightarrow \mathfrak{J}(\Sigma)$ concretely as follows:

$$\iota(z,x) = \left(z, z, \ldots, z, \frac{N\beta}{2\pi i} x \right). \tag{3.4}$$

From our discussion above, the subspace $\mathfrak{J}_0$ consists of the orthogonal subspace to this embedding $\mathfrak{J}(\Sigma) = \mathfrak{Q} \oplus \mathfrak{J}_0$. The phase space of the reduced system $\mathbf{Y}$, the fibration of $\mathfrak{J}_0$ over $\mathbf{P}^{N-1}$, is then bi-rationally equivalent to the space of $N-1$ points in $\mathfrak{Q}$ (the relative coordinates of N particles) with local coordinates $Z_i = (q_i, p_i)$. The relation between the action/angle variables of the reduced system $\{a_i\}$ and $\{\theta_i\}$ and the $N-1$ points in $\mathfrak{Q}$ mirrors the discussion above for the four-dimensional case. First of all, we fix the overall position of Σ in $\mathfrak{Q}$. Then, at least generically, given $N-1$ points in $\mathfrak{Q}$ there exists a unique curve $\Sigma \subset \mathfrak{Q}$ containing these points. If P_j, $j = 1, \ldots, N-1$, are these points on Σ then we may obtain a point in the reduced jacobian $\mathfrak{J}_0$ by first mapping into the jacobian $\mathfrak{J}(\Sigma)$

via the Abel map:

$$\theta_i = \sum_{j=1}^{N-1} \int_{P_0}^{P_j} \omega_i \,, \tag{3.5}$$

where P_0 is a fixed base point, and then by using the projection $\mathfrak{J}(\Sigma) \to \mathfrak{J}_0$. Under this equivalence, the symplectic form is preserved so that[4]

$$\sum_{i=1}^{N+1} \delta a_i \wedge \delta\theta_i = \sum_{i=1}^{N-1} \delta p_i \wedge \delta q_i \,. \tag{3.6}$$

Here, the one-forms $\{\delta\theta_i\}$ are constrained to lie in the reduced jacobian $\mathfrak{J}_0$ so that the number of independent degrees-of-freedom match on both sides. The right-hand side of (3.6) just corresponds to $N-1$ copies of the standard symplectic form on $\mathfrak{Q}$.

We now prove (3.6). Suppose we choose z to be a local coordinate on Σ. The curve would be described by the holomorphic function $x = x(z)$. Consider the abelian differentials of the first kind on Σ, ω_i, $i = 1, \ldots, N+1$. Explicitly in this coordinate system $\omega_i = f_i(z)dz$ and (3.5) becomes

$$\theta_i = \sum_{j=1}^{N-1} \int_{q_0}^{q_j} f_i(z) dz \,. \tag{3.7}$$

Now suppose we vary the Z_i. The point in the jacobian will vary as

$$\delta\theta_i = \sum_{j=1}^{N-1} f_i(q_j)\, \delta q_j \,. \tag{3.8}$$

Now we turn to the action variables. How do we describe these? As we vary the actions we change our curve Σ and this will be described by the normal bundle $N_\Sigma = T\mathfrak{Q}|_\Sigma / T\Sigma$ to Σ in $\mathfrak{Q}$. Then, using the holomorphic symplectic form, $T\mathfrak{Q} \cong T^\star\mathfrak{Q}$ and consequently $N_\Sigma \cong T^\star\Sigma$. Thus we may identify our actions with holomorphic sections of $T^\star\Sigma$,

$$x(z)\, dz = \sum_{j=1}^{N+1} a_j\, \omega_j(z) \,. \tag{3.9}$$

Using the contour integrals around the 1-cycles A_j associated to the basis ω_j:

$$a_j = \oint_{A_j} x\, dz \,, \tag{3.10}$$

where on the right-hand side a pull-back from $\mathfrak{Q}$ to Σ is implied. Then $x\,dz$ plays the role of the Seiberg-Witten differential. Hence under a variation at the point with coordinate z

$$\sum_{j=1}^{N+1} \delta a_j\, \omega_j(z) = \delta x(z)\, dz \,. \tag{3.11}$$

[4]Using δ rather than d for differential forms on the phase space, avoids confusion with differential forms on Σ.

Now we evaluate this at $z = q_i$ where $\delta x(q_i) = \delta p_i$, giving

$$\delta p_i = \sum_{j=1}^{N+1} \delta a_j f_j(q_i)\,. \tag{3.12}$$

Using first (3.12) and then (3.8), the result (3.6) follows:

$$\sum_{i=1}^{N-1} \delta p_i \wedge \delta q_i = \sum_{i=1}^{N-1}\sum_{j=1}^{N+1} (f_j(q_i)\delta a_j)\wedge \delta q_i = \sum_{i=1}^{N-1}\sum_{j=1}^{N+1} \delta a_j \wedge (f_j(q_i)\delta q_i) = \sum_{j=1}^{N+1} \delta a_j \wedge \delta\theta_j\,. \tag{3.13}$$

Note that the right-hand side only depends on $N-1$ independent variations $\{\delta\theta_j\}$.

3.1 Example: the $N = 2$ system

We will describe how this picture arises in the case of $N = 2$. This case is rather simple because the integrable system describing the relative motion is only one-dimensional. Here our curve Σ has genus 3 and we have that $\mathfrak{J}(\Sigma)$ is isogenous to $\mathfrak{Q} \oplus \mathfrak{J}_0$ where now $\mathfrak{J}_0$ corresponds to an elliptic curve. Thus our reduced phase space $\mathbf{Y}$ consists of an elliptic curve fibred over $\mathfrak{M}_0 \equiv \mathbf{P}^1$. This is in fact a Kummer K3 surface, which alternately appears as the fibre $\pi^{-1}(0)$ of the projection $\pi:\ \mathrm{Hilb}^{[2]}\mathfrak{Q} \to \mathrm{Sym}^2(\mathfrak{Q}) \to \mathfrak{Q}$ which comes from using the group structure on $\mathfrak{Q}$.

Thus for the $N = 2$ theory, $\mathfrak{M}_0$ is described by a single modulus which we parameterize by $H = -A_{00}/A_{01}$. In this case, we can map the reduced system to a single point in $\mathfrak{Q}$ with local coordinate (q, p). According to the discussion in the last section the curve whose overall position in $\mathfrak{Q}$ is determined by requiring that it goes through the point (q, p). It follows trivially that

$$H(q,p) = \frac{\Theta\begin{bmatrix}0 & 1/2\\ 0 & 0\end{bmatrix}\left(q \quad \beta/\pi i p\,\middle|\,\begin{matrix}\tau & \beta m/\pi i\\ \beta m/\pi i & 2\rho\end{matrix}\right)}{\Theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\left(q \quad \beta/\pi i p\,\middle|\,\begin{matrix}\tau & \beta m/\pi i\\ \beta m/\pi i & 2\rho\end{matrix}\right)}. \tag{3.14}$$

We now show that this hamiltonian reduces to the hamiltonian of the centre-of-mass motion of the 2-body elliptic Ruijsenaars-Schneider system and then the elliptic Calogero-Moser system on decompactification of $\tilde{\mathbf{T}}^2$ in one and then two directions, respectively.

Let us take the five-dimensional limit by taking the limit $\rho \to i\infty$. In this limit, up to a simple re-scaling, we have

$$H_{5\mathrm{d}}(q,p) = e^{\beta p}\frac{\Theta\begin{bmatrix}0\\0\end{bmatrix}(q + \beta m/2\pi i|\tau)}{\Theta\begin{bmatrix}0\\0\end{bmatrix}(q|\tau)} + e^{-\beta p}\frac{\Theta\begin{bmatrix}0\\0\end{bmatrix}(q - \beta m/2\pi i|\tau)}{\Theta\begin{bmatrix}0\\0\end{bmatrix}(q|\tau)}. \tag{3.15}$$

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In order to reproduce the conventional hamiltonian of the centre-of-mass motion of the 2-body elliptic Ruijsenaars-Schneider system, we must shift

$$p \to p - \frac{1}{2\beta} \log \frac{\Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(q + \beta m / 2\pi i | \tau\right)}{\Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(q - \beta m / 2\pi i | \tau\right)}, \qquad q \to q + \frac{1}{2}(1 + \tau). \tag{3.16}$$

In which case, up to another simple re-scaling,

$$H_{5\text{d}}(q,p) = \cosh(\beta p) \sqrt{\wp\left(\frac{\beta m}{2\pi i}\right) - \wp(q)}, \tag{3.17}$$

where $\wp(q)$ is the Weierstrass function with periods 1 and τ.

The hamiltonian appropriate to the four-dimensional theory is obtained by taking $\beta \to 0$ which yields, again up to a simple re-scaling,

$$H_{4\text{d}}(q,p) = p^2 + \frac{m^2}{2\pi^2} \wp(q), \tag{3.18}$$

which is the well-known hamiltonian of the two-particle elliptic Calogero-Moser system.

It remains to discuss the connection between the integrable system of this paper and that of the Dell system [12]. (We shall use the notation of the latter paper when making comparison.) Both approaches involve a genus three curve in a $(1,2)$ polarized abelian variety: here we had $\Sigma \hookrightarrow \mathfrak{Q}$, while for the Dell system this was $\bar{\mathcal{C}}_z \hookrightarrow S$. Also, in both approaches, after removing the center of mass coordinates from $\mathfrak{J}(\Sigma)$ (respectively $\mathfrak{J}(\bar{\mathcal{C}}_z)$) we are left with (for the $N = 2$ case) a one-dimensional abelian variety, or elliptic curve, here $\mathfrak{J}_0$ and there $\pi^\star(E_z)$. (Further, both works give a construction involving Prym varieties.) The hamiltonians constructed in these different approaches look rather different and it remains to connect them more directly. This will be left for a later work [36]. The advantage of the approach adopted in the present paper is that it immediately generalizes to $N > 2$, a generalization that was previously unknown.

4. Discussion

The present paper has shown how to construct the Seiberg-Witten curve of the six-dimensional $\mathcal{N} = (1,1)$ gauge theory compactified on a torus to four dimensions. The argument given was based on Witten's M-theory construction of $\mathcal{N} = 2$ theories but the curve we arrive may be naturally constructed from various approaches [4, 6]. Central to our discussion was the appearance of a $(1,N)$ polarized abelian variety $\mathfrak{Q}$ specified by the complex structures τ, ρ and mass m. In addition to the four-dimensional coordinates of the M5-brane, this was specified by a holomorphic curve Σ embedded in $\mathfrak{Q}$, of genus $N+1$: the Seiberg-Witten curve. We presented this curve explicitly in various forms via theta functions showing agreement with the curve of [4] describing N instantons in a U(1) theory on a non-commutative torus. The quiver generalization to a $\mathrm{U}(N)^k$ gauge theory, associated to N instantons in $\mathrm{U}(k)$, will simply change the polarization of the abelian variety under consideration.

We also have demonstrated how the curve Σ may be viewed as the spectral curve of an integrable system which generalizes the N-body elliptic Calogero-Moser and Ruijsenaars-Schneider systems associated to four and five-dimensional gauge theories. The resulting system is not simply doubly elliptic as the positions and momenta, as two-vectors, take values in the ambient space $\mathfrak{Q}$. The integrable system we have constructed is rather natural: the curve Σ in $\mathfrak{Q}$ is described by the linear system $\mathbf{P}(H^0(\mathfrak{Q},\mathcal{O}(\mathcal{L}))^\star) \cong \mathbf{P}^{N-1}$, our space of reduced actions, together with a translation in $\mathfrak{Q}$, our "centre-of-mass" coordinates. Together with our actions $\mathcal{J}(\Sigma)$ these form a $2(N+1)$ dimensional phase space which may be understood as a particular example of a class of integrable systems discovered by Mukai [37, 38] who constructed the symplectic structure on the moduli space of stable sheaves on a symplectic surface (a particular example being our abelian surface $\mathfrak{Q}$). If one considers the moduli space $\mathcal{M}_{\mathfrak{Q}}(0,c_1,c_2)$ of sheaves on $\mathfrak{Q}$ of rank 0, $c_1 = c_1(\mathcal{L})$ and some c_2 then one gets a symplectic moduli space, which is a relative jacobian (i.e., union of jacobians). The fact that the rank is zero here means these are torsion sheaves and $c_1 = c_1(\mathcal{L})$ says that the support of the sheaves is a curve, which is cohomologous to our curve Σ. Our phase space is bi-rational to this moduli space which itself is bi-rational to the product of the Hilbert scheme $\mathrm{Hilb}^{[N]}\mathfrak{Q}$ with $\mathfrak{Q}$. We exhibited the canonical transformation between the action-angle variables and the set of $N-1$ points in $\mathfrak{Q}$ corresponding to the reduced phase space. Finally, although we showed that our construction has many features in common with the known Dell system for two particles, we have yet to relate the two rather different descriptions of the hamiltonians. We shall return to this and other features of our integrable system in [36].

Another interesting issue regarding the new integrable system is whether it admits a Lax-type representation. One would expect that such a representation should follow from an associated Hitchin system. The Hitchin system should follow from a D-brane construction involving the four torus $\mathfrak{Q}$, as in [4, 33, 34]. This will be investigated elsewhere.

To conclude, the construction of our paper has been premised on a fixed polarized abelian variety $\mathfrak{Q}$ specified by the complex structures τ, ρ and mass m. In general we could consider each such theory over the moduli space of such polarized abelian varieties. The moduli space of polarized abelian varieties is itself well studied [39, 40] and we observe that there is a natural $\mathrm{sl}(2,\mathbb{Z}) \times \mathrm{sl}(2,\mathbb{Z})$ acting on this. It is interesting to ask how string theory sweeps out this moduli space.

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